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An Artin relation of length n in the braid group

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Abstract

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In Coxeter (1984) the existence of some Artin type relations in unitary reflection groups was examined which do not appear explicitly among the standard defining relations. This paper examines a particularly nice occurrence in the braid groups, and comprises a solution to Problem 100, posed in Coxeter (1989).

Let $A \overset{q}{\leftrightarrow} B$ denote the relation $ABABA\cdots = BABAB\cdots$ with q letters on each side of the equals sign. Thus, $A \overset{1}{\leftrightarrow} B$ means $A = B$, and $A \overset{2r}{\longleftrightarrow} B$ means $(AB)^r = (BA)^r$. In this notation, the braid group on $n+1$ strings, $\mathscr{B}(n+1)$, with generators $R_1, R_2, \ldots, R_n$, has the presentation

$$R_\mu \overset{3}{\leftrightarrow} R_{\mu+1} \quad \text{and} \quad R_\mu \overset{2}{\leftrightarrow} R_\nu, \quad (\nu > \mu + 1).$$

Set $P = R_1R_3R_5\cdots$ and $Q = R_2R_4R_6\cdots$. We show that P and Q satisfy $P \overset{n+1}{\longleftrightarrow} Q$.

Let ρ denote the *reversing automorphism* of $\mathscr{B}(n+1)$ defined by $\rho(R_i) = R_{n+1-i}$. Also if g is an element of a group, let α_g denote the inner automorphism corresponding to g, i.e., $\alpha_g(x) = g^{-1}xg$. Define $C_1 = R_1$, $C_j = R_j^{-1}R_{j-1}R_j$ for $2 \leqslant j \leqslant n$, and $C_{n+1} = R_n$.

Notice that the relation $R_{j-1} \overset{3}{\leftrightarrow} R_j$ implies not only $R_{j-1}R_jR_{j-1} = R_jR_{j-1}R_j$ but also $R_{j-1}R_jR_{j-1}^{-1} = R_j^{-1}R_{j-1}R_j$.

Lemma 1. *The subgroup generated by $R_1, \ldots, R_j$ (for $j \leqslant n$) is equally well generated by $C_1, \ldots, C_j$.*

Proof. This is trivial for $j = 1$. If $j > 1$, then we have

$$R_{j-1}R_jR_{j-1}^{-1} = R_j^{-1}R_{j-1}R_j,$$

$$\begin{tikzcd}
C_1 \arrow[r, "\alpha_Q"] & C_2 \arrow[r, "\alpha_P"] & C_3 \arrow[r] & \cdots \arrow[r] & C_n \arrow[r, "\alpha_Q"] & C_{n+1} \arrow[d, "\alpha_P"] \\
& & \text{for } n \text{ odd,} & & & \\
\rho(C_{n+1}) \arrow[uu, "\alpha_P"] & \rho(C_n) \arrow[l, "\alpha_Q"] & \rho(C_{n-1}) \arrow[l, "\alpha_P"] & \cdots \arrow[l] & \rho(C_2) \arrow[l] & \rho(C_1) \arrow[l, "\alpha_Q"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
C_1 \arrow[r, "\alpha_Q"] & C_2 \arrow[r, "\alpha_P"] & C_3 \arrow[r] & \cdots \arrow[r] & C_n \arrow[r, "\alpha_P"] & C_{n+1} \arrow[d, "\alpha_Q"] \\
& & \text{for } n \text{ even.} & & & \\
\rho(C_{n+1}) \arrow[uu, "\alpha_P"] & \rho(C_n) \arrow[l, "\alpha_Q"] & \rho(C_{n-1}) \arrow[l, "\alpha_P"] & \cdots \arrow[l] & \rho(C_2) \arrow[l] & \rho(C_1) \arrow[l, "\alpha_P"]
\end{tikzcd}$$

Fig. 1.

whence $R_j = R_{j-1}^{-1} C_j R_{j-1}$, and it follows by induction the R_j is expressible in terms of $C_1, \ldots, C_j$. $\square$

Lemma 2. *The diagrams in Fig.* 1 *describe the orbit of* C_1 *under* α_P *and* α_Q *alternately.*

Proof. We need only check the top and right sides of the diagrams, as the other two sides will then follow from the relations $\rho\alpha_P = \alpha_Q\rho$, for n even, and $\rho\alpha_P = \alpha_P\rho$, and $\rho\alpha_Q = \alpha_Q\rho$, for n odd.

$$C_1 Q = R_1 R_2 R_4 R_6 \cdots = R_4 R_6 \cdots R_1 R_2 = R_2 R_4 R_6 \cdots R_2^{-1} R_1 R_2 = Q C_2,$$

and for $j \geqslant 1$ we have

$$\begin{aligned}
C_{2j+1} Q &= (R_{2j} R_{2j+1} R_{2j}^{-1})(R_2 R_4 \cdots R_{2j} R_{2j+2} \cdots) \\
&= R_2 R_4 \cdots R_{2j-2} R_{2j} R_{2j+1} R_{2j+2} \cdots \\
&= R_2 R_4 \cdots R_{2j} R_{2j+2} \cdots R_{2j+2}^{-1} R_{2j+1} R_{2j+2} \\
&= Q C_{2j+2}
\end{aligned}$$

and

$$\begin{aligned}
C_{2j} P &= (R_{2j+1} R_{2j} R_{2j+1}^{-1})(R_1 R_3 \cdots R_{2j-1} R_{2j+1} \cdots) \\
&= R_1 R_3 \cdots R_{2j-3} R_{2j-1} R_{2j} R_{2j+1} \cdots \\
&= R_1 R_3 \cdots R_{2j-1} R_{2j+1} \cdots R_{2j+1}^{-1} R_{2j} R_{2j+1} \\
&= Q C_{2j+1}.
\end{aligned}$$

Also if n is even, $C_{n+1} Q = R_n R_2 R_4 \cdots R_n = Q R_n = Q\rho(C_1)$ and if n is odd, then $C_{n+1} P = R_n R_1 R_3 \cdots R_n = P R_n = P\rho(C_1)$. $\square$

We may now prove the following.

Theorem 1. $P \overset{n+1}{\longleftrightarrow} Q$.

Proof. Setting $S = QPQP \cdots$ with $n+1$ factors, we see from Lemma 2 that for $k \geq 0$, $\alpha_S(C_{2k+1}) = \rho(C_{2k+1})$ and $\alpha_S(\rho(C_{n-2k})) = C_{n-2k}$. Thus the action of α_S is to reverse the braids $\{C_1, \rho(C_n), C_3, \rho(C_{n-2}), C_4, \ldots\}$, which by Lemma 1 generate $\mathscr{B}(n+1)$, and so $\alpha_S = \rho$. If n is even, then $\alpha_S(P) = \rho(P) = Q$, and so $PS = SQ$, $S = QP \cdots PQ = PSQ^{-1} = PQ \cdots QP$ and $P \overset{n+1}{\longleftrightarrow} Q$. If n is odd, then $\alpha_S(P) = \rho(P) = P$, so $QP \cdots QP = SP = PS = PQ \cdots PQ$ and again $P \overset{n+1}{\longleftrightarrow} Q$. $\square$

Corollary. *$S^2 = \Delta$, the unique positive word generating the infinite cyclic center of $\mathscr{B}(n+1)$, which represents a full twist of the straight braid by* 360°. *S itself is represented by a twist of* 180°.

Proof. $\alpha_S^2 = \alpha_{S^2}$ is the identity, so S^2 is in the center of $\mathscr{B}(n+1)$. It is the generator $(R_1R_2 \cdots R_n)^{n+1}$ since S^2 is a positive word of the correct length; see [2]. To see that S is indeed a 180° twist, assume that the endpoints of the strings are placed, instead of along a line, on the circumference of a circle, as in Fig. 2, and turn slowly. $\square$

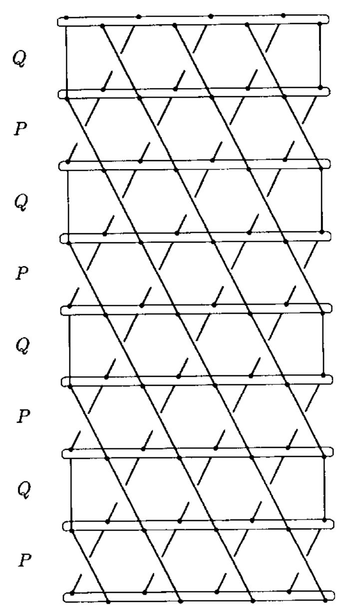

Fig. 2. The element S in B(6).

Given an edge weighted graph Γ, there is an associated *Artin group* presented by taking the vertices of Γ as generators, and relations $v_i \overset{k}{\leftrightarrow} v_j$ if vertices v_i and v_j are joined by an edge of weight k; see [1]. The braid group is an example of an Artin group, and the Coxeter groups are obtained by requiring additionally that the square of every generator is 1. If all the weights in Γ are greater than 1, it is unknown if two non-isomorphic graphs can present isomorphic Artin groups.

Consider the set of elements $P_{2j+1} = R_1 R_3 \cdots R_{2j+1}$ and $Q_{2i} = R_2 R_4 \cdots R_{2i}$, $2i+1,\ 2j < n$. These n elements form a generating set of $\mathscr{B}(n+1)$ and we have seen that between any pair there is an Artin type relation;

$$P_i \overset{2}{\leftrightarrow} P_j \qquad Q_i \overset{2}{\leftrightarrow} Q_j$$

$$P_i \overset{i+1}{\longleftrightarrow} Q_j \qquad Q_i \overset{i+1}{\longleftrightarrow} P_j, \qquad \text{for } i<j.$$

It is thus pertinent to ask if these generators and relations constitute a presentation of $\mathscr{B}(n+1)$. The answer, however, is no, even in the smallest interesting case, that of the 4 string braid group generated by R_1, R_2 and R_3. To see this, let $a = P_1 = R_1$, $b = Q_2 = R_2$, $c = P_3 = R_1 R_3$, and let $\mathscr{G}$ denote the group with presentation

$$\langle a, b, c \mid a \overset{3}{\leftrightarrow} b,\ b \overset{4}{\leftrightarrow} c,\ a \overset{2}{\leftrightarrow} c \rangle.$$

Define $\mathscr{G}^*$ to be $\mathscr{G}$ with the additional relations $a^2 = b^2 = c^2 = 1$. It is easy to see that $\mathscr{G}^*$ is a finite group of order 48, namely the complete symmetry group of the cube. Its Cayley graph is illustrated in Fig. 3. For a different drawing of the same graph, see [6, p. 76].

However, if the group so presented were $\mathscr{B}(4)$, then $\mathscr{G}^*$ would be the quotient of $\mathscr{B}(4)$ by the relations $R_1^2 = R_2^2 = R_3^2 = 1$, which yields the symmetric group on 4 letters, having half as many elements as $\mathscr{G}^*$.

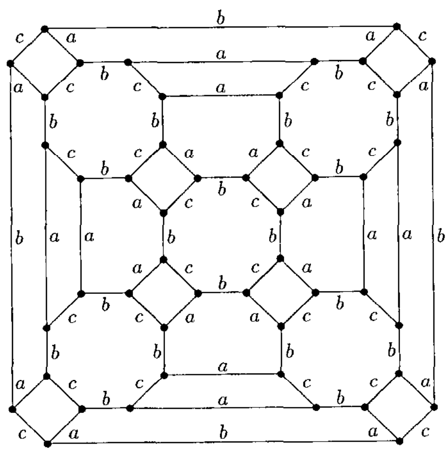

Fig. 3. The rhombitruncated cuboctahedron.

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